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EXISTENCE AND STABILITY OF A PERIODIC SOLUTION FOR AN EQUATION GOVERNING DYNAMICS OF A RENEWABLE RESOURCE SUBJECTED TO ADDITIVE ALLEE EFFECTS

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Abstract. Allee effect refers to reduction in individual fitness at low population densities. Among several Allee effects that are known to occur in species dynamics, we consider additive Allee effect which appear to exist in several vital systems in the real world. In this article we study the dynamics of a renewable resource that is subjected to additive Allee effects in a periodically varying environment. We derive conditions under which the considered model admits a positive periodic solution. Uniqueness and stability properties of this periodic solution are also investigated. Fixed point theory and comparison principle have been employed to establish the needed results. It is observed that the trivial solution of the considered model is unstable where as the positive periodic solution is asymptotically stable and attracts all other solutions with positive initial states. This study highlights the conditions under which the stock in a renewable resource that is influenced by additive Allee effect follows a stable periodic pattern eventually where the periodicity coincides with that of the environment.

Key Words. Renewable resource, Periodic solution, uniqueness, stability, additive Allee effect

AMS(MOS) subject classification. 34C25, 92D25

1. Introduction.

The term Allee effect seems to have originated from the works of Allee [3, 4]. Allee effect refers to reduction in individual fitness at low population density that can lead to extinction [6, 9, 10]. These effects are strongly related to the extinction vulnerability of populations. A few mechanisms generating Allee effects have been highlighted in [6, 10].

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In this work we are interested in studying the consequences of presence of *additive Allee effect* [1, 2, 12] (among various Allee effects that are known to exist) on the dynamics of a renewable resource in a periodic environment. This additive Allee effect refers to reduction of species due to an extra mortality rate influenced by factors such as satiation of a predator [10, 7, 14, 13, 25] anti predator behaviour like group defence against predator and inhibition [15, 24, 26, 30] or the necessity to find a mate for reproduction [7, 13, 20] etc. Two interesting derivations for the term representing additive Allee effect can be found in [25] which are developed in the context of search for a mate and impact of a satiating generalist predator.

Studying the consequence of Allee effects on the dynamics of a renewable resource under the influence of seasonal variations defines a vital problem as a real world application. Some of the recent works associated with renewable resource dynamics considered in seasonally varying environment include [8, 23]. Bio-economics of a renewable resource in a seasonally varying environment has been presented in [8]. Leggett-William multiple fixed point theorem has been applied in [23] to obtain sufficient conditions for the existence of at least two positive periodic solutions for a differential equation that governs dynamics of a renewable resource subjected to strong Allee effects (multiplicative Allee effects) in seasonal varying environments.

Several well known fixed point theorems are used in [17, 27] to establish non existence of periodic solutions and existence of single and multiple periodic solutions for periodic functional differential equations. [28, 29] present results pertaining to existence of periodic solutions for singular differential equations and [21, 22] present results for existence of periodic solutions for ordinary differential systems. In this article we concentrate on the existence and stability aspects of a unique positive periodic solution for a differential equation that governs the dynamics of a renewable resource subjected to additive Allee effects in a periodic environment. The existence of a positive periodic solutions is established using a fixed point theorem.

Describing species dynamics using periodic differential equations enables us to study the influence of seasonal variations on the species of interest. Thus periodicity plays an important role in the problems associated with real world that are influenced by seasonal variations. In the process of analyzing the consequences of such periodic variations in the environment it is reasonable, as a first approximation, to consider the involved parameters to be periodic of the same period. Thus a natural approach might then be to study the effects of periodic variations in the appropriate parameters of the model equations which have been used to describe the growth dynamics in constant environments [8, 23, 11].

In the next section we introduce the model. In section 3 the existence of at least one positive periodic solution for a general scalar differential equation has been examined. In section 4 we investigate the existence of a unique asymptotically stable positive periodic solution for the general equation to which the considered model becomes a special case. Section 5 illustrates the existence and asymptotic stability of a unique periodic solution through numerical simulation. This is followed by discussion in section 6.

2. The Model. Let us consider the following differential equation representing the dynamics of a renewable resource subjected to additive Allee effect [1, 2, 25]:

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{K} - \frac{\nu}{1+\omega x}\right) \tag{1}$$

where the positive constants r, K respectively represent intrinsic growth rate and carrying capacity of the resource ν, ω indicate the severity of Allee effect that has been modeled [1, 2]. Considering these parameters to be positive constants, it can be easily observed that the equation (1) always admits the trivial solution as one of its equilibrium solutions and it admits at the most two positive equilibrium solutions depending on the values of the involved parameters. Assuming that the parameters of (1) are constants imply that r, K and the severity of the Allee effect do not change with time and are independent of seasons. Alternatively, incorporating periodicity in the coefficients of the model enables one to study the dynamics of the resource in a seasonally varying environment. In this article we are interested in studying the qualitative behaviour of (1) under the assumption that the associated parameters are periodic of same period, P. Hence we assume the coefficients of (1) to be non negative P-periodic continuous functions given by $r(\tau)$, $K(\tau)$, $\nu(\tau)$ and $\omega(\tau)$. Under this assumption (1) gets modified to

$$\frac{dx}{d\tau} = r(\tau)x\left(1 - \frac{x}{K(\tau)} - \frac{\nu(\tau)}{1+\omega(\tau)x}\right) \tag{2}$$

The following lemma transforms the P-periodic differential equation (2) in to another equivalent similar (involving only three periodic coefficients instead of four) T-periodic differential equation where T could be different from P.

LEMMA 1. *The transformation* $t = G(\tau) = \int_0^\tau r(s)ds$ *transforms the (2) to a* T *- periodic equation given by*

$$\frac{dy}{dt} = y\left(1 - \frac{y}{k(t)} - \frac{\eta(t)}{1 + m(t)y}\right) \tag{3}$$

with $y(t) = x(G^{-1}(t))$ *where* $k(t) = K(G^{-1}(t),\ \eta(t) = \nu(G^{-1}(t))$ and $m(t) = \omega(G^{-1}(t))$ *are positive periodic functions of period* $T = G(P)$. *Also, for each* T *- periodic solution* $y(t)$ *of (3)* $x(\tau) = x(G^{-1}(t))$ *defines a* P *- periodic solution of (2).*

In the light of the above Lemma 1 we shall concentrate on the existence of a T-periodic solution for (3)

3. Existence of at least one periodic solution. In this section we shall study the existence of at least one positive periodic solution for a fairly general scalar differential equation. The results developed in this section will be applied to (3) to find conditions that guarantee the existence of at least one positive $T-$periodic solution.

Let us consider the following scalar differential equation

$$\frac{dy}{dt} = y - f(t, y) \tag{4}$$

where f, defined on $R \times R$, is a non negative real valued continuous function in both the arguments and satisfies $f(t + T, y) = f(t, y),\ T > 0$. The following two lemmas can be easily established.

LEMMA 2. $y(t)$ *is a* $T-$ *periodic solution of (4) if and only if it satisfies the integral equation*

$$y(t) = \int_t^{t+T} G(t, s)f(s, y(s))ds \tag{5}$$

where $G(t, s)$ *is the Green's function given by*

$$G(t, s) = \frac{e^{-(s-t)}}{1 - e^{-T}},\ s \in [t, t + T].$$

LEMMA 3. *The function* $G(t, s)$ *satisfies*

(i) $G(t + T, s + T) = G(t, s)$ for $s \in [t, t + T]$

(ii) $\frac{e^{-T}}{1-e^{-T}} = G(t, t + T) \le G(t, s) \le G(t, t) = \frac{1}{1-e^{-T}}$

(iii) $1 \le \frac{G(t,s)}{G(t,t+T)} \le e^T$ for any $s \in [t, t + T]$

(iv) $\int\limits_t^{t+T} G(t,s)ds = 1.$

The following definition and Lemma are needed for the analysis to follow.

DEFINITION 1. *[19] Let Y be a Banach space. A non empty closed subset $K \subset Y$ is called a cone if the following conditions are satisfied:*

i) *If $y \in K$, then $\lambda y \in K$ for $\lambda \geq 0$,*

ii) *If $y \in K$ and $-y \in K$ then $y = 0$*

LEMMA 4. *[31] Let Y be a Banach space and let $K \subset Y$ be a cone in Y. Assume Ω_1, Ω_2 are open subsets of Y with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $\phi : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a complectly continuous operator satisfying any one of the following hypotheses H_1 or H_2.*

H_1 *i* $\|\phi y\| \leq \|y\|$ for all $y \in K \cap \partial\Omega_1$

ii there exist $\psi \in K\backslash\{0\}$ *such that* $y \neq \phi y + \lambda\psi$ for $y \in K \cap \partial\Omega_2$ and $\lambda > 0$

H_2 *iii* $\|\phi y\| \leq \|y\|$ for all $y \in K \cap \partial\Omega_2$

iv there exists $\psi \in K\backslash\{0\}$ such that $y \neq \phi y + \lambda\psi$ for $y \in K \cap \partial\Omega_1$ and $\lambda > 0$.

Then ϕ has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let us consider the following Banach space

$$Y = \{y : y \in C(R,R), y(t+T) = y(t)\}$$

endowed with the norm $\|y\| = \sup\limits_{0 \leq t \leq T} y(t)$ and the cone

$$K = \{y \in Y : y(t) \geq 0 \text{ and } y(t) \geq e^{-T}\|y\|\}.$$

Define an operator $\phi : Y \to Y$ by

$$(\phi y)(t) = \int\limits_t^{t+T} G(t,s) f(s, y(s)) ds. \tag{6}$$

It can be easily verified that ϕ is a completely continuous operator.

LEMMA 5. $\phi(K) \subset K$

Proof. In view of non negativity of $f(t,y)$, for any $y \in K$ we have

$$(\phi y)(t) = \int\limits_t^{t+T} G(t,s) f(s,y(s)) ds = \int\limits_t^{t+T} \frac{e^{-(s-t)}}{1-e^{-T}} f(s,y(s)) ds \geq 0. \tag{7}$$

Using Lemma 3 we have

$$\|\phi y\| = \sup_{0\le t\le T} \int_t^{t+T} G(t,s)f(s,y(s))ds \le \frac{1}{1-e^{-T}} \int_t^{t+T} f(s,y(s))ds$$

and

$$\begin{aligned}(\phi y)(t) = \int_t^{t+T} G(t,s)f(s,y(s))ds &\ge \frac{e^{-T}}{1-e^{-T}} \int_t^{t+T} f(s,y(s))ds \\ &\ge e^{-T}\|\phi y\|.\end{aligned}$$

Therefore

$$(\phi y)(t) \ge e^{-T}\|\phi y\| \tag{8}$$

Hence, from (7) and (8), we have $\phi y \in K$ for any $y \in K$. This completes the proof. ∎

THEOREM 1. *Assume that*

(H_3) $\limsup_{y\to 0} \max_{0\le t\le T} \left\{\frac{f(t,y)}{y}\right\} < 1$

and

(H_4) $\liminf_{y\to\infty} \min_{0\le t\le T} \left\{\frac{f(t,y)}{y}\right\} > 1.$

Then (4) admits at least one positive T-periodic solution.

Proof. From H_3 there exists a positive constant r_1 such that

$$f(t,y) \le y \quad \text{for } 0 < |y| \le r_1. \tag{9}$$

For any $y \in K$ with $\|y\| = r_1$ we have

$$\begin{aligned}\|\phi y\| = \sup_{0\le t\le T}\int_t^{t+T} G(t,s)f(s,y(s))ds \ &\le \ \int_t^{t+T} G(t,s)y(s)ds \text{ (from (9))} \\ &\le \ \int_t^{t+T} G(t,s)\|y\|ds \\ &= \ \|y\|\int_t^{t+T} G(t,s)ds \\ &= \ \|y\|.\end{aligned}$$

Therefore

$$\|\phi y\| \le \|y\| \text{ for } y \in K \cap \partial\Omega_1 \tag{10}$$

where $\Omega_1 = \{y \in Y : \|y\| < r_1\}$. Below we shall establish the validity of the second condition of Lemma 4. From H_4 there exists a positive constant ρ such that

$$f(t,y) \ge y \quad \text{for } |y| > \rho. \tag{11}$$

Let $\Omega_2 = \{y \in Y : \|y\| \ < r_2\}$ with $e^{-T}r_2 > \rho$. Then for any $y \in K \cap \partial\Omega_2,\ y(t) \ge e^{-T}\|y\|$. We claim that

$$y \ne \phi y + \psi\lambda$$

for $\psi = 1$ and for $y \in K \cap \partial\Omega_2$ with $\lambda > 0$.

Let if possible there exists a $y_0 \in K \cap \partial\Omega_2$ and $\lambda_0 > 0$ such that $y_0 = \phi y_0 + \lambda_0$

and $\mu = \min\limits_{0 \le t \le T} y_0(t)$. We have

$$\begin{aligned} y_0(t) = (\phi y_0)(t) + \lambda_0 &= \int\limits_t^{t+T} G(t,s) f(s, y_0(s)) ds + \lambda_0 \\ &\geq \int\limits_t^{t+T} G(t,s) y_0(s) ds + \lambda_0 \quad \text{(from (11))} \\ &\geq \mu \int\limits_t^{t+T} G(t,s) ds + \lambda_0 \\ &= \mu + \lambda_0. \end{aligned}$$

This implies that $\mu \geq \mu + \lambda_0$, a contradiction. Thus, from Lemma 4, it follows that ϕ has a fixed point $y^* \in K \cap (\overline{\Omega_2} \backslash \Omega_1)$ which is a positive T-periodic solution of (4) that satisfies $r_1 < y^* < r_2$. ∎

The following theorem is an application of Theorem 1 for the existence of at least one positive T-periodic solution for (3).

THEOREM 2. *If* $\max\limits_{0 \le t \le T} \{\eta(t)\} < 1$ *then (3) admits at least one positive* T*-periodic solution.*

4. Uniqueness of positive periodic solution. In this section we develop results that establish uniqueness of the positive T - periodic solution for (3) and study its stability properties. Hence forth we assume that $\max\limits_{0 \le t \le T} \{\eta(t)\} < 1$ which ensures existence of at least one positive $T-$ periodic solution for (3) (Theorem 2).

Since the coefficient functions $k(t)$, $\eta(t)$ and $m(t)$ of (3) are positive T-periodic continuous functions they are bounded and hence there exist positive constants k_1, k_2, a, b, c and d satisfying $k_1 \le k(t) \le k_2$, $a \le \eta(t) \le b < 1$ and $c \le m(t) \le d$. Clearly we have

$$y\left(1 - \frac{y}{k_1} - \frac{b}{1+cy}\right) \le y\left(1 - \frac{y}{k(t)} - \frac{\eta(t)}{1+m(t)y}\right) \le y\left(1 - \frac{y}{k_2} - \frac{a}{1+dy}\right) \tag{12}$$

Now, let us consider the differential equations

$$\frac{dy}{dt} = yF(t,y) \tag{13}$$

$$\frac{dy}{dt} = yG(y) \tag{14}$$

and

$$\frac{dy}{dt} = yH(y) \tag{15}$$

where

$$F(t,y) = 1 - \frac{y}{k(t)} - \frac{\eta(t)}{1+m(t)y}$$
$$G(y) = 1 - \frac{y}{k_1} - \frac{b}{1+cy}$$
$$H(y) = 1 - \frac{y}{k_2} - \frac{a}{1+dy}$$

From (12) and for $y > 0$ we have

$$yG(y) \le yF(t,y) \le yH(y) \tag{16}$$

The autonomous nature of (14) and (15) make it possible to study their equilibrium solutions and analyze their stability nature. Observe that each of the equations (14) and (15) admit the trivial equilibrium solution $y = 0$ and a positive equilibrium solution given by

$$y_G = \frac{ck_1 - 1 + \sqrt{(ck_1-1)^2 + 4ck_1(1-b)}}{2c}$$

and

$$y_H = \frac{dk_2 - 1 + \sqrt{(dk_2-1)^2 + 4dk_2(1-a)}}{2d}.$$

respectively. From the qualitative behaviour of $G(y)$ and $H(y)$ the trivial solutions are unstable where as y_G and y_H are asymptotically stable. In view of relation (16) we have $y_G < y_H$. We need the following two theorems for the analysis to follow.

THEOREM 3. *[16] If a solution $\varphi(t,0,y_0)$ of a T-periodic differential equation $\dot{y} = f(t,y)$ is bounded for $t \ge 0$, then there is a T-periodic solution $\Phi(t)$ of equation $\dot{y} = f(t,y)$ such that*

$$\varphi(t+T\kappa,0,y_0) \to \Phi(t) \quad \text{as the integer} \quad \kappa \to \infty$$

monotonically and uniformly for $0 \le t \le T$. Similarly, if $\varphi(t,0,y_0)$ is bounded for $t \le 0$, then there is a T-periodic solution $\Psi(t)$ of equation

$\dot{y} = f(t, y)$ *such that*

$$\varphi(t - T\kappa, 0, y_0) \to \Psi(t) \text{ as the integer } \kappa \to \infty$$

monotonically and uniformly for $0 \le t \le T$.

THEOREM 4. *Let* $\dot{y} = yf(t, y)$ *be a* T*-periodic differential equation. Let there exist two functions* $g(y)$ *and* $h(y)$ *such that* $g(y) < f(t, y) < h(y)$ *for all* y *and for all* t *with* $g(0) > 0$. *Let* $\tilde{y}$ *and* $\tilde{\tilde{y}}$ *be a unique set of positive constants satisfying* $g(\tilde{y}) = 0$, $h(\tilde{\tilde{y}}) = 0$. *Further, let* $f(t, y)$ *be decreasing in* y *for* $y > 0$. *Then*

(a) *if* $y(0) > 0$ *then* $y(t) > 0$ *for all* t.

(b) *there is a unique positive* T*-periodic solution* $y^*(t)$ *of the considered differential equation satisfying* $\tilde{y} < y^*(t) < \tilde{\tilde{y}}$.

(c) *all solutions with positive initial value approach the periodic solution* $y^*(t)$ *asymptotically.*

Proof. **(a)** Follows from an application of the comparison principle [18] to the equations $\dot{y} = yg(y)$ and $\dot{y} = yf(t, y)$.

(b) Clearly $\tilde{y}$ and $\tilde{\tilde{y}}$ are equilibrium solutions of $\dot{y} = yg(y)$ and $\dot{y} = yh(y)$ respectively. Applying comparison principle [18] to the equations $\dot{y} = yg(y)$, $\dot{y} = yf(t, y)$ and $\dot{y} = yh(y)$ it can be easily observed that a solution $y(t)$ of $\dot{y} = yf(t, y)$ with $\tilde{y} < y(0) < \tilde{\tilde{y}}$ satisfies $\tilde{y} < y(t) < \tilde{\tilde{y}}$. Application of Theorem 3 yields the existence of a positive $T-$periodic solution $y^*(t)$ that satisfies $\tilde{y} < y^*(t) < \tilde{\tilde{y}}$. Now we shall show that the equation $\dot{y} = yf(t, y)$ admits a unique T-periodic solution bounded below and above by $\tilde{y}$ and $\tilde{\tilde{y}}$ respectively. For, let if possible, $v(t)$ and $u(t)$ be two distinct positive periodic solutions of $\dot{y} = yf(t, y)$ such that $\tilde{y} < u(t) < v(t) < \tilde{\tilde{y}}$. Define $\omega(t) = v(t) - u(t) > 0$. Since $v(t)$ and $u(t)$ are periodic in nature so is $\omega(t)$. Differentiating $\omega(t) = v(t) - u(t)$ with respect to t we obtain

$$\begin{aligned} \dot{\omega}(t) = \dot{v}(t) - \dot{u}(t) &= v(t)f(t, v(t)) - u(t)f(t, u(t)) \\ &= v(t)(f(t, v(t)) - f(t, u(t))) + \omega(t)f(t, u(t)). \end{aligned} \tag{17}$$

Since $f(t, y)$ is decreasing for $y > 0$ we have

$$f(t, v(t)) - f(t, u(t)) < 0. \tag{18}$$

From (17) and (18) we obtain

$$\dot{\omega}(t) < \omega(t)f(t, u(t)). \tag{19}$$

Hence we have $\omega(T) < \omega(0)$ which contradicts the fact that $\omega(t)$ is T- peri-

odic. Hence $v(t) = u(t)$.
(c) Let $y_h(t)$ be a solution of $\dot{y} = yh(y)$. If $y_h(0) > \tilde{\tilde{y}}$ then $y_h(t)$ approaches $\tilde{\tilde{y}}$ asymptotically and monotonically. Therefore

$$\lim_{t\to\infty} y_h(t) = \tilde{\tilde{y}}.$$

From an application of the comparison principle [18] we can observe that a solution $y(t)$ of $\dot{y} = yf(t,y)$ with initial value $y_h(0)$ satisfies

$$y(t) < y_h(t) \text{ for all } t > 0. \tag{20}$$

We claim that there exists a $\tau > 0$, sufficiently large, such that $y(t) < \tilde{\tilde{y}}$ for all $t \geq \tau$. Let if possible $y(t) > \tilde{\tilde{y}}$ for all $t > 0$. Since $y_h(t)$ approaches $\tilde{\tilde{y}}$ monotonically, it follows that $y(t) > y_h(t)$ for some sufficiently large t. This contradicts (20). Hence the claim.
Similarly if $0 < y(0) < \tilde{y}$ then there exists a $\bar{\tau} > 0$, sufficiently large such that $y(t) > \tilde{y}$ for all $t \geq \bar{\tau}$. Therefore, all positive solutions of $y' = yf(t,y)$ satisfy $\tilde{y} < y(t) < \tilde{\tilde{y}}$ for sufficiently large t. Thus, by Theorem 3, all solutions of $y' = yf(t,y)$ with positive initial values approach the unique T-periodic solution $y^*(t)$ asymptotically. ∎

The following is an application of Theorem 4 for the existence of unique positive periodic solution of (3).
THEOREM 5. *If $b < 1$ and $bdk_2 < 1$ then (3) admits a unique positive asymptotically stable T-periodic solution $y^*(t)$ satisfying $y_G < y^*(t) < y_H$.*

Proof. Clearly $G(y_G) = 0$, $H(y_H) = 0$ and $G(0) = 1 - b > 0$. It is easy to verify that $\frac{\partial F}{\partial y} < 0$ for $y > 0$ if $bdk_2 < 1$. Hence, by Theorem 4, the equation (3) admits a unique positive asymptotically stable T-periodic solution $y^*(t)$ satisfying $y_G < y^*(t) < y_H$. ∎

5. Numerical illustration. In this section we illustrate the existence of unique asymptotically stable positive periodic solution for (2). Let us consider the following model that describes dynamics of a fishery influenced by additive Allee effect in periodically fluctuating environment:

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{K - \sigma_k sin(\frac{2\pi\tau}{cycle})} - \frac{\eta - \sigma_\eta sin(\frac{2\pi\tau}{cycle})}{1 + (m - \sigma_m sin(\frac{2\pi\tau}{cycle}))x}\right). \tag{21}$$

Forms for some of the coefficient functions are adopted from [5]. Significance of various parameters considered in (21) and their values are presented

TABLE 1
Table presenting the description of various parameters along with their values and units pertaining to (21)

Parameter	Description	Value	Units
r	Intrinsic growth rate	0.36	1/year
K	Carrying capacity	3.5	million ton
σ_K	Amplitude of K fluctuation	1.5	million ton
η	parameter inducing additive Allee effect	0.8	-
σ_η	Amplitude of η fluctuation	0.1	-
m	parameter inducing additive Allee effect	0.1	1/million ton
σ_m	Amplitude of m fluctuation	0.1	1/million ton
cycle	Environmental cycle	50	years

in the Table 1. Values for some of the parameters are taken from the parameter set considered in [5]. Clearly (21) is a $50 - year$ periodic differential equation. We shall employ the theory developed in the previous section to check if (21) admits a periodic solution. An application of Lemma 1 transforms (21) to the following equivalent $18 -$ periodic differential equation:

$$\frac{dx}{dt} = x\left(1 - \frac{x}{K - \sigma_k sin(\frac{2\pi t}{18})} - \frac{\eta - \sigma_\eta sin(\frac{2\pi t}{18})}{1 + (m - \sigma_m sin(\frac{2\pi t}{18}))x}\right). \tag{22}$$

It is easy to observe that the choice of parameters as in Table 1 satisfy the conditions of the Theorem 5 and hence the equation (22) admits a unique asymptotically stable $18 -$ periodic solution. Fig. 1 presents the said periodic solution and also illustrates its asymptotic stability nature by showing the asymptotic approach of two solutions with distinct initial states. Hence (21) admits a unique $50 - year$ periodic asymptotically stable positive periodic solution.

Comparing the Theorems 2 and 5 we observe that although the restriction on the function $\eta(t)$ that its maximum be less than unity assures existence of at least one positive periodic solution for (3), restriction on the magnitude of either the variable carrying capacity $K(t)$ or on the Allee effect inducing periodic function $m(t)$ or both is needed for ensuring the existence of a unique positive periodic solution which is asymptotically stable. However numerical experiments indicate that the condition in the Theorem 2 alone guarantees existence of a unique asymptotically stable positive periodic solution. Fig. 2 gives the unique asymptotically stable positive periodic solution of (22) when m and σ_m are taken to be 25 and 10 respectively with

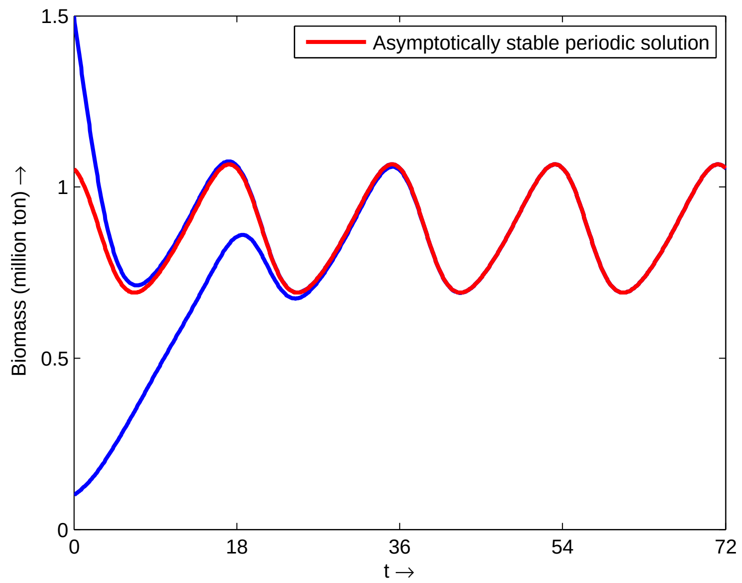

FIG. 1. *This figure presents the unique positive periodic solution (red colour curve) admitted by the equation (22) with the choice of coefficient functions as given in the Table 1). The behaviour of other solutions (blue colour curves) with positive initial values demonstrate asymptotic stability nature of the unique positive periodic solution of the equation (22)*

all other parameter values as in Table 1. Observe that these parameter values satisfy Theorem 2 but not Theorem 5. These numerical experiments indicate the necessity to improve the existence result that guarantees existence of exactly one positive periodic solution which is asymptotically stable.

6. Discussion and conclusions. Growth of renewable resources such as fisheries or other biotic species heavily depend on seasonal variations which are basically periodic in nature. Thus following the dynamics of renewable resources in a periodic environment takes the study closer to reality. In this context we have considered a differential equation representing the dynamics of renewable resource subjected to additive Allee effect in a periodic environment. Our interest is to investigate if the considered equation admits any periodic solutions at all. If it admits such solutions what would be their stability nature.

To start with, a fairly general first order differential equation has been considered and conditions are established for this equation to admit at least one positive periodic solution. Fixed point theory techniques have been used

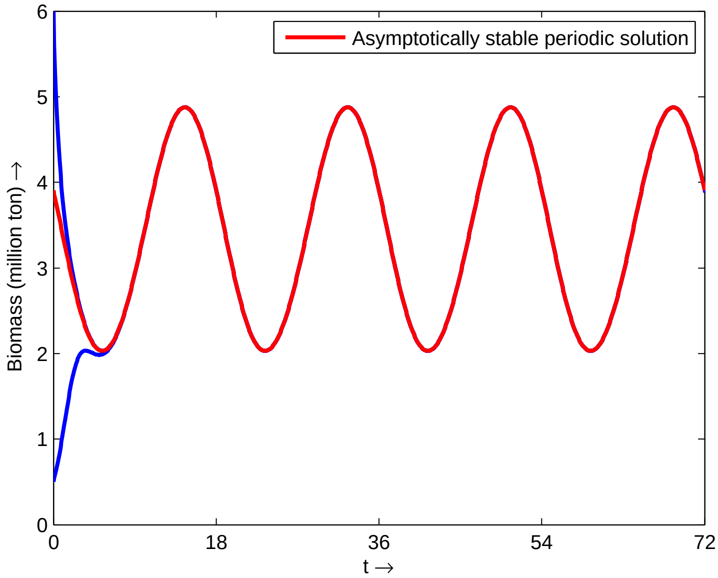

FIG. 2. *This figure presents the asymptotic stability nature of the unique positive periodic solution (red colour curve) admitted by the equation (22) when the coefficient functions satisfy Theorem 2 and do not satisfy Theorem 5, indicating a necessity to establish results that relax the condition ($bdk_2 < 1$) of Theorem 5. The blue curves represent solutions of (22) with positive initial state whose asymptotic approach to the periodic solution demonstrates the asymptotic stable behaviour of the periodic solution*

to conclude these results. Then the considered model for the renewable resource has been viewed as a special case for the general equation and sufficient conditions are derived that promise existence of at least one positive periodic solution.

It is very essential to know the exact number of periodic solutions admitted by a model representing resource dynamics and be aware of their stability nature in order to derive strategies for optimal use of the natural resources. Thus results have been derived that ensure existence of a unique positive periodic solution which attracts all other solutions with positive initial values. Comparision principle has been used to obtain these results. The key findings are illustrated through numerical simulation.

Finally we conclude that a renewable resource that is subjected to additive Allee effects and influenced by seasonal variations eventually evolves to a state that exhibits periodicity if the magnitude of the factors that are responsible for the occurrence of Allee effects in the system is sufficiently small. The numerical simulations indicate a necessity to improve the existence results that ensure asymptotic stability of unique positive periodic solution. The results developed in this article form a firm base for studies on Bio-economics of renewable resources in a seasonally varying environment [8].

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